

## Synchronization of extended systems from internal coherence

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A condition for the synchronizability of a pair of extended systems governed by partial differential equations (PDEs), coupled through a finite set of variables, is commonly the existence of internal synchronization or internal coherence in each system separately. The condition was previously illustrated in a forced-dissipative system and is here extended to Hamiltonian systems using an example from particle physics. Full synchronization is precluded by Liouville's theorem. A form of synchronization weaker than "measure synchronization" is manifest as the positional coincidence of coherent oscillations ("breathers" or "oscillons") in a pair of coupled scalar field models in an expanding universe with a nonlinear potential, and does not occur with a variant of the model that does not exhibit oscillons.

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The phenomenon of synchronized chaos, initially explored in low-order ordinary-differential-equation (ODE) systems [1–3,8], has been extended to partial differential equations (PDEs) that describe a variety of systems of physical interest [4]. Chaos synchronization extends the paradigm of synchronization of regular oscillators that is ubiquitous in nature [5]. One seeks an understanding of the internal properties of a chaotic physical system that will allow a pair of such systems, loosely coupled, to synchronize, despite sensitive dependence on initial conditions. Spatially extended systems offer richer possibilities for relationships that fall short of full synchronization than do ODE systems. In a geophysical example previously studied [6], it was seen that slaving of small scales, a relationship that defines an *inertial manifold*, was crucial to synchronizability. Here, it is suggested that more general interscale relationships, as may give rise to coherent structures within each system separately, are required for the synchronizability of the pair. The connection is illustrated in a particle physics context: a toy model of a scalar field in the expanding early universe [7], a Hamiltonian system without an attractor. Weak scale relationships allow oscillons (breathers) to persist and give rise to a new form of synchronization defined by the coincidence in position of oscillons in two coupled scalar field models. Conversely, where the dynamics do not admit such coherent structures, vestiges of synchronization are lost.

In the non-Hamiltonian meteorological example described in [6], which we next review, two geophysical fluid systems, representing planetary-scale wind patterns, are coupled only through their medium-scale Fourier components. Each system is given by a potential vorticity equation

$$\frac{Dq_i}{Dt} \equiv \frac{\partial q_i}{\partial t} + J(\psi_i, q_i) = F_i + D_i, \quad (1)$$

where the stream function  $\psi$  is the fundamental dynamical variable, the Jacobian  $J(\psi, \cdot) = \frac{\partial \psi}{\partial x} \frac{\partial \cdot}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \cdot}{\partial x}$  gives the advective contribution to the Lagrangian derivative  $D/Dt$ , there are two horizontal layers  $i=1,2$ , and the potential vorticity  $q$ , which generalizes angular momentum, is a derived variable

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defined in terms of  $\psi$  in [6]. Potential vorticity is conserved on a moving parcel, except for forcing  $F_i$  and dissipation  $D_i$ .

Two models of form (1),  $Dq^A/Dt = F^A + D^A$  and  $Dq^B/Dt = F^B + D^B$  were coupled diffusively through one of the forcing terms:

$$\begin{aligned} F_{\vec{k}}^A &= \mu_{\vec{k}}^c [q_{\vec{k}}^B - q_{\vec{k}}^A] + \mu_{\vec{k}}^{\text{ext}} [q_{\vec{k}}^* - q_{\vec{k}}^A], \\ F_{\vec{k}}^B &= \mu_{\vec{k}}^c [q_{\vec{k}}^A - q_{\vec{k}}^B] + \mu_{\vec{k}}^{\text{ext}} [q_{\vec{k}}^* - q_{\vec{k}}^B], \end{aligned} \quad (2)$$

where the flow has been decomposed spectrally and the subscript  $\vec{k}$  on each quantity indicates the wave-number  $\vec{k}$  spectral component (suppressing the index  $i$ ). A background flow  $q^*$  forces each system separately. The set of coefficients  $\mu_{\vec{k}}^c$  was chosen to couple the two channels only in some medium range of wave numbers. Band-limited coupling defined by  $\mu_{\vec{k}}^c$  replaces the coupling of two PDE systems at a discrete set of points as in [4].

The two systems synchronize over time in Fig. 1, where the contours of  $\psi$  are streamlines that define the flow. The synchronization is manifest as the coincidence of structures of meteorological significance ("blocking patterns" that interrupt the flow) in both space and time. That coincidence is

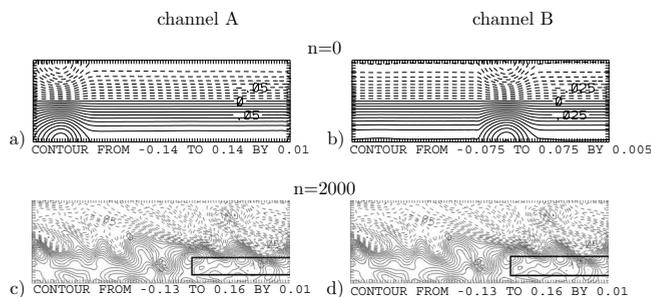


FIG. 1. Stream function  $\psi$  (in units of  $1.48 \times 10^9 \text{ m}^2 \text{ s}^{-1}$ , averaged over layers  $i=1,2$ ) describing the flow at [(a) and (b)] initial and [(c) and (d)] final times, in a parallel channel model with coupling of medium scale modes for which  $|k_x| > k_{x0}=3$  or  $|k_y| > k_{y0}=2$ , and  $|k| \leq 15$  for the indicated numbers  $n$  of time steps in a numerical integration. Parameters are as in [6]. Synchronization occurs by the last time shown, in (c) and (d), despite differing initial conditions. The blocking patterns in the boxed areas coincide [9].

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robust against significant differences in the two systems [6].

The large scales need not be coupled because of well-known dynamical relations between scales (“inverse cascade”) in 2D turbulence [9]. But the stronger result [6] that synchronization occurs without coupling of the smallest scales is explained simply: the smallest-scale components are thought not to be independent dynamical variables but are functions of the medium- and large-scale components, defining a dynamically invariant *inertial manifold*. Where the set of variables that are coupled is smaller than a minimal independent set, synchronization does not occur. Partial synchronization of forced-dissipative systems will usually be easy to arrange, as previously found by Kocarev *et al.* [4], since *approximate inertial manifolds* (AIMs) exist for almost all parabolic PDEs [10].

In Hamiltonian systems, the subject of the current work, inertial manifolds cannot exist since the stability of such a manifold would imply a collapse of phase space volumes for trajectories that start off the manifold, contradicting Liouville’s theorem. A simple example is the Klein-Gordon equation in an expanding background geometry, in one space and one time dimension with cyclic boundary conditions, with a nonlinear potential of a type that gives rise to oscillons, possibly representing the first coherent structures in the universe [7]. The field satisfies

$$\frac{\partial^2 \phi}{\partial t^2} + H \frac{\partial \phi}{\partial t} - e^{-2Ht} \frac{\partial^2 \phi}{\partial x^2} + V'(\phi) = 0, \quad (3)$$

where  $H$  is a Hubble constant, the potential  $V$  is given by  $V(\phi) = (1/2)\phi^2 - (1/4)\phi^4 + (1/6)\phi^6$  for units in which the scalar field mass  $m = 1$ , and the prime denotes the derivative with respect to  $\phi$ . Equation (3) is derived by replacing derivatives in the standard Klein-Gordon equation by covariant derivatives for a Robertson-Walker metric describing expansion with Hubble constant  $H$  [11], giving a term in  $\partial\phi/\partial t$  that formally resembles friction but which in fact preserves the Hamiltonian structure. Indeed, system (3) is derivable from a time-dependent Hamiltonian density

$$\mathcal{H} = (1/2)e^{-Ht}[(\phi_x)^2 + \pi^2] + e^{Ht}V(\phi), \quad (4)$$

where  $\pi \equiv e^{Ht}\dot{\phi}$  is the canonical momentum that is conjugate to  $\phi$ . Subscripts here denote partial derivatives. Liouville’s theorem applies even with the time dependence so that neither an inertial manifold, in the strict sense, nor an AIM with a usefully small approximation bound can exist.

Oscillating coherent structures appear as regions of high-energy density at final time, in the numerical integration shown in Fig. 2(a), after initializing with thermal noise as in [7], with zero-point quantum fluctuations also represented.

Consider two systems [Eq. (3)] coupled diffusively, through some Fourier components of the field only, according to

$$\frac{\partial^2 \phi^{A,B}}{\partial t^2} + H \frac{\partial \phi^{A,B}}{\partial t} - e^{-2Ht} \frac{\partial^2 \phi^{A,B}}{\partial x^2} + V'(\phi^{A,B}) - F^{A,B} = 0, \quad (5)$$

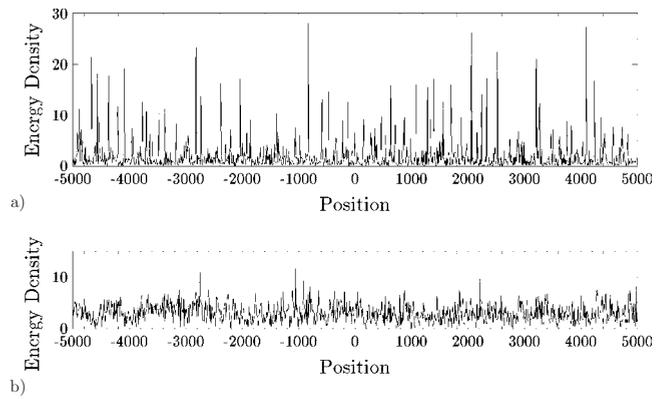


FIG. 2. Energy density  $\rho = (1/2)e^{-Ht}(\phi_x)^2 + (1/2)e^{Ht}(\phi_t)^2 + e^{Ht}V(\phi)$  vs position  $x$  for (a) a numerical simulation of Eq. (3) suggesting localized oscillons and (b) a simulation of the same equation but with a different potential  $V(\phi) = (1/2)\phi^2 + (1/4)\phi^4 + (1/6)\phi^6$  for which oscillons do not occur, shown for comparison.

$$F_k^A = c_k \left( \frac{\partial \phi_k^B}{\partial t} - \frac{\partial \phi_k^A}{\partial t} \right) \quad F_k^B = c_k \left( \frac{\partial \phi_k^A}{\partial t} - \frac{\partial \phi_k^B}{\partial t} \right) \quad (6)$$

with the coupling coefficients  $c_k = 0$  vanishing above a threshold value  $|k| > k_0$  and set to a large value for  $|k| \leq k_0$  so that corresponding large scale components in the two systems are effectively clamped. For coupling of the lowest wave-number modes, up to wavelengths of about twice the final oscillon width, oscillons in the two systems were found to occur at mostly the same locations though their amplitudes differed, as shown in Fig. 3(a). In contrast, if the two systems were left completely uncoupled, but shared initial conditions over a range of scales, and only the smallest scales, well below the oscillon widths, were initialized differently, then oscillons formed at locations that were uncorrelated [Fig. 3(b)]. Apparently, there is a “butterfly effect,” as in meteorology, through which the small scales have a large impact on the positions of formation of coherent structures [12]. But the dynamical evolution of these structures then partially slaves a portion of the small-scale sector (as with shock waves) and proceeds independently of the remaining portion.

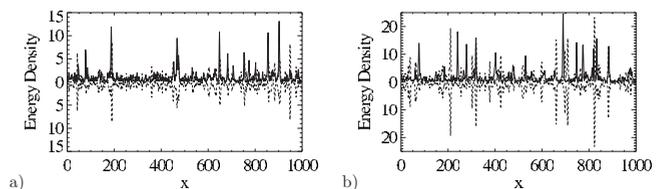


FIG. 3. (a) The local energy density  $\rho$  vs  $x$  for two concurrent simulations (solid and dashed lines, resp.) of the oscillon system [Eq. (3)], coupled according to Eqs. (5) and (6), with coupling coefficient  $c_k = 2$  for  $k \leq 64$  and  $c_k = 0$  otherwise, at final time. [ $\rho$  for the second system (dashed line) is also shown reflected across the  $x$  axis for ease in comparison.] The coincidence of oscillon positions is apparent; (b) local energy density  $\rho$  vs  $x$  at final time for two simulations of the oscillon system, plotted as in (a), but with common initialization of all modes with  $k < 0.8 \times 2^{14}$ , and no subsequent coupling between the two systems. Oscillon positions appear uncorrelated.

With severely attenuated initial noise [Fig. 4(a)], oscillon synchronization occurs with an even narrower range of coupled wave numbers. The large-scale components ( $k \leq 32$ ) of the same oscillons are shown in the bottom portion of Fig. 4(b). For four of the five oscillons shown, the oscillon positions are coincident with the local maxima of the truncated field. For the remaining one (the leftmost), the oscillon is displaced from the local maximum of the truncated field but is still coincident with the corresponding oscillon in the other system [Fig. 4(a)]. The small-scale Fourier components, in both situations, are slaved to the large-scale components insofar as they determine oscillon position.

Coincidence of oscillon positions suggests measure synchronization, the weak form of synchronized chaos in which the trajectories of two coupled systems become the same when the systems are coupled, without a requirement that the states of the systems are the same at a given instant of time [13,14]. Measure synchronization is characteristic of jointly Hamiltonian coupled systems. Here, the coupled oscillon systems do not quite attain measure synchronization since the corresponding oscillons differ in amplitude, and the configuration [Eq. (5)], which can be written as

$$\dot{\phi}_k^{A,B} = e^{-Ht} \pi_k = \partial \mathcal{H}^{A,B} / \partial \pi_k^{A,B}, \quad (7)$$

$$\dot{\pi}_k^{A,B} = -\partial \mathcal{H}^{A,B} / \partial \phi_k^{A,B} + c_k(\pi_k^{B,A} - \pi_k^{A,B}), \quad (8)$$

is not derivable from a joint Hamiltonian. [It would be Hamiltonian if the second term in Eq. (8) were to be replaced by  $c_k(\phi_k^{B,A} - \phi_k^{A,B})$ .] But even without the joint Hamiltonian structure, complete synchronization is not likely achievable by coupling a finite number of modes if the remaining infinite number are not slaved.

In a comparable system without oscillons, there are no vestiges of synchronization. Correlations between corresponding modes in the coupled systems are displayed in (a) of Table I for the oscillon system, and in (b) of Table I for an alternate pair of systems with potential  $V = V_{\text{alt}}(\phi) = (1/2)\phi^2 + (1/4)\phi^4 + (1/6)\phi^6$  in Eq. (3), for which no oscillons form, as seen in Fig. 2(b). (Oscillons occur in one case and not the other because the  $-\phi^4$  term in the first case gives a flatter potential so that larger amplitude oscillations have lower frequencies and decouple from the faster and smaller traveling

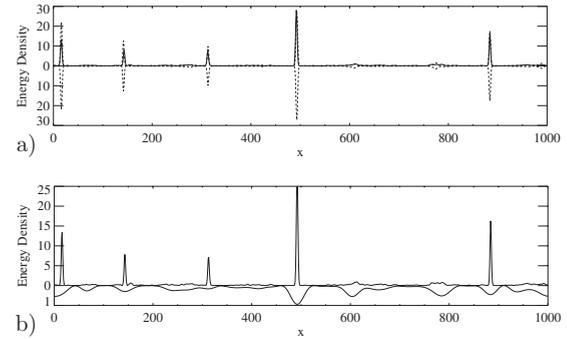


FIG. 4. (a) The local energy density  $\rho$  vs  $x$  for two simulations of the oscillon system [Eq. (3)], coupled according to Eqs. (5) and (6), with coupling coefficient  $c_k=2$  for  $k \leq 32$  and  $c_k=0$  otherwise, at final time, displayed as in Fig. 3. The initial noise level was severely attenuated as compared to the thermal initialization used in the simulation in Fig. 3: the amplitude of the  $n$ th Fourier component at initial time was multiplied by  $(1/n)^{0.35}$ . (b) Local energy density  $\rho$  vs  $x$  at final time for one of the two simulations shown in panel (a), with the part of  $\rho$  corresponding to  $k \leq 32$  shown reflected in the negative- $y$  portion of the panel (where  $y$  is rescaled for clarity).

wave solutions that would otherwise cause them to dissipate.) While there are small but significant correlations between some of the uncoupled modes in (a) of Table I, corresponding to the small partially slaved portion of the small-scale sector, no significant correlations appear between corresponding small-scale components that are not coupled in (b) of Table I. The correlations in the case with oscillons are expected to extend to much shorter wavelengths in longer simulations that approach the oscillon lifetimes as the oscillons decrease in width (in comoving coordinates) and background fluctuations decrease in amplitude [7].

The synchronization results for oscillons suggest a unifying principle governing synchronization in both Hamiltonian and forced-dissipative systems: two complex systems of either type can be made to synchronize, with a restricted set of coupled variables, if and only if each system exhibits synchronization internally. Sufficiency of the condition follows from transitivity if the internal synchronization is exact: any external coupling that does not destroy the internal relationships and causes some pairs of variables in the two systems to agree will also cause all variables that internally synchro-

TABLE I. Correlations between coefficients of corresponding Fourier components of the independent dynamical variables in (a) the  $A$  and  $B$  subsystems of the coupled scalar field system [Eq. (5)] and in (b) the same system with the potential  $V_{\text{alt}}$ . The Fourier coefficients, indexed by  $n$ , are partitioned into coupled modes with wave number  $k=[(n+1)/2]$ ,  $0 \leq n < 128$  (in cycles/domain length), and several ranges of uncoupled modes. Error bars for the first range of uncoupled modes are at  $2\sigma$ , where  $\sigma$  was computed as the standard error of the mean based on a further partitioning into odd and even  $n$ . Uncoupled modes correlate significantly only with a nonlinear potential that supports oscillons.

	(a) Correlations with $V = \phi^2/2 - \phi^4/4 + \phi^6/6$		(b) Correlations with $V_{\text{alt}} = \phi^2/2 + \phi^4/4 + \phi^6/6$	
	$\phi$	$\partial\phi/\partial t$	$\phi$	$\partial\phi/\partial t$
Coupled modes: $0 \leq n < 128$	1.00	1.00	1.00	1.00
Uncoupled modes: $128 \leq n < 256$	$0.31 \pm 0.04$	$0.18 \pm 0.18$	$-0.06 \pm 0.04$	$0.17 \pm 0.18$
Uncoupled modes: $256 \leq n < 384$	0.08	0.00	0.06	-0.14
Uncoupled modes: $384 \leq n < 512$	0.02	0.04	0.04	0.00
All uncoupled modes: $128 \leq n < 16258$	0.08	0.00	0.00	0.01

nize with the coupled variables to agree with their counterparts in the other system. In the cases considered here, the condition also appears necessary. Localized coherent structures such as oscillons, as well as long-range connections such as the Atlantic-Pacific “teleconnection” described previously [6,9], can be regarded as approximate forms of internal synchronization. Where a master set of variables, in Fourier space, synchronizes with the entire remaining infinite set, an inertial manifold exists. Local coherence has been described as synchronization of a more limited type in PDEs [15] and in lattices of coupled maps [16].

The internal synchronization condition also provides guidance as to the choice of coupled variables. If the extended systems are “truth” and a “computer model” to which truth is coupled in one direction only, via observations, as in meteorology [17], then a set of variables should be observed that is synchronized with a maximal set of other variables internally.

An internal coherence criterion applies to ODE systems, consistently with Pecora and Carroll’s criterion of negative “conditional Lyapunov exponents” in the Lorenz system [1]. Lorenz  $X$  can be said to approximately synchronize with Lorenz  $Y$ , along the near-planar Lorenz attractor, in agreement with the well-known fact that sufficiently strong coupling of either  $X$  or  $Y$  in one system to the corresponding variable of a second Lorenz system will synchronize the two systems. Coupling of the  $Z$  variables that do not correlate internally with either  $X$  or  $Y$  will not do so. But the internal coherence criterion is much more useful for PDEs that describe spatially extended systems both because such systems are less tractable analytically—a full set of conditional Lyapunov exponents is hard to compute—and because the coherent structures are more meaningful physically.

For Hamiltonian systems coherent structures seem to bear strongly on synchronizability and on the form of synchronization. The ubiquity of coherent structures in solutions to nonlinear PDEs is the basis on which potentially broad relevance is claimed for the current work. Solitons, for instance, in a pair of PDE systems that exhibit them, might also be made to coincide in position and to move in syn-

chrony if only a restricted set of corresponding Fourier components of the two fields are coupled. The question is whether the internal dynamics that allow the structures to exist in each system would cause the needed parts of the uncoupled components to follow. The phenomenon, if it exists, might be used for secure communications in the same manner as the systems considered in [4].

The present generalization differs from measure synchronization [13] in several notable respects: first, in an ergodic system with trajectories that define a uniform measure, such as the system with modified potential  $V_{\text{alt}}$  that exhibits no oscillon behavior, measure synchronization is trivial. Second, as already pointed out, positional coincidence of coherent structures that differ in amplitude or detailed shape is more general than measure synchronization. Third, there is no requirement that the *combined* system be Hamiltonian to exhibit the weak form of synchronization described here. The impossibility of strict synchronization follows not from the joint Hamiltonian structure but from the impossibility of slaving or approximately slaving all uncoupled variables in each system separately.

The types of synchronization appear to be equivalent, in their details, to detailed interscale relationships within each system internally, which in the present case allow oscillons to persist and to stably maintain their positions. The partial agreement of small scales between the two synchronized systems, sufficient to force coincidence in oscillon position when large scales are clamped, indeed defines a partial slaving that counters the butterfly effect. For consistency with Liouville’s Theorem, there must be a compensating expansion of the remaining part of the uncoupled modes in phase space, as entropy is cast off to scales that are yet smaller. Details of the partial slaving remain to be worked out. The existence and the form of synchronization may provide diagnostics for the slaving relationships.

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